Lezione 6

venerdì 20 novembre 2020 09:56

SVILUPPI DEI CONCETTI DI CONVERGENZA PUNTUALE E UNIFORME DI SUCCESSIONI DI FUNZIONI

$f_n : (a,b) = I \longrightarrow \mathbb{R} \qquad I \subset \mathbb{R}$

$n \in \mathbb{N}$

$$\underbrace{\bullet\ f_n \xrightarrow{I} f \quad \text{se } \forall x \in I, \forall \varepsilon > 0\ \exists \bar{n} = \bar{n}(\varepsilon, x) \in \mathbb{N} \text{ tale che } \forall n > \bar{n} \implies |f_n(x) - f(x)| < \varepsilon}_{\text{CONVERGENZA PUNTUALE IN } I}$$

$$\underbrace{\bullet\ f_n \overset{I}{\rightrightarrows} f \quad \text{se } \forall \varepsilon > 0\ \exists \bar{n} = \underset{\mathbb{N}}{\bar{n}(\varepsilon)} \text{ tale che } \forall n > \bar{n} \implies |f_n(x) - f(x)| < \varepsilon \ \forall x \in I}_{\text{CONVERGENZA UNIFORME IN } I}$$

la convergenza uniforme in $I \implies$ converg puntuale in I

IL VICEVERSA È FALSO

$(\varepsilon, x) \longrightarrow \bar{n}(\varepsilon, x) \qquad \sup_{x \in I} \bar{n}(\varepsilon, x) = \bar{n}(\varepsilon) \in \mathbb{N}\ (\neq +\infty)$

$\sup_{x \in I}$

$\uparrow$ $\mathbb{N}$ ($\neq +\infty$)

CONV. UNIF IN I

TEOREMA (criterio per la convergenza uniforme)

Sia $(f_n)_n$ una successione di funzioni

$f_n \; I = (a,b) \longrightarrow \mathbb{R}$

Supponiamo che $f_n \xrightarrow{I} f$, $\quad f \; I \longrightarrow \mathbb{R}$

Sia $L_n = \sup_{x \in I} |f_n(x) - f(x)|$: dunque L_n è una succ.

di numeri reali

$$f_n \xrightarrow{I} f \iff L_n \longrightarrow 0 \quad (n \longrightarrow \infty)$$

Dimostrazione

- "$L_n \longrightarrow 0 \implies f_n \overset{I}{\rightrightarrows} f$" (da dimostrare)

Se $\forall \varepsilon' > 0 \; \exists \; \bar{n} = \bar{n}(\varepsilon) : \forall n > \bar{n} \implies |L_n| < \varepsilon$

$\Downarrow$?

$\forall \varepsilon > 0 \; \exists \; \tilde{n} = \tilde{n}(\varepsilon)$ t.c. se

$\forall n > \tilde{n} \implies |f_n(x) - f(x)| < \varepsilon \quad \forall x \in I$

$\varepsilon = \varepsilon' \longrightarrow \bar{n} = \bar{n}(\varepsilon) : |L_n| < \varepsilon$

L_n

$\varepsilon > L_n = \sup_{x \in I} |f_n(x) - f(x)| \geq |f_n(x) - f(x)| \quad \forall x \in I$

$\tilde{n} = \bar{n} \longrightarrow f_n \overset{I}{\rightrightarrows} f$

- $f_n \overset{I}{\rightrightarrows} f \quad \forall \varepsilon > 0 \ \exists \tilde{n} = \tilde{n}(\varepsilon) \text{ t.c. } \forall n > \tilde{n}$ $\left.\begin{matrix} \\ \end{matrix}\right\}$ Hp
  $\Rightarrow |f_n(x) - f(x)| < \varepsilon \quad \forall x \in I$

(da dim) $\forall \varepsilon' \quad \exists \bar{n} = \bar{n}(\varepsilon') : \forall n > \bar{n} \Rightarrow |L_n| = L_n < \varepsilon'$

$\varepsilon' \longrightarrow \varepsilon = \frac{\varepsilon'}{2} \longrightarrow \tilde{n}\left(\frac{\varepsilon'}{2}\right) . \forall n > \tilde{n} \Rightarrow |f_n(x) - f(x)| < \frac{\varepsilon'}{2}$

$\forall x \in I$

$\Rightarrow \sup_{x \in I} |f_n(x) - f(x)| = L_n \leq \frac{\varepsilon'}{2} < \varepsilon'$

$\bar{n}(\varepsilon') = \tilde{n}\left(\frac{\varepsilon'}{2}\right) \Rightarrow$ ovvero $L_n \longrightarrow 0$.

ES.

$f_n(x) = \frac{x^{2n}}{1 + x^{2n}} \qquad x \in \mathbb{R}$

CONV. PUNT.

se fissato in $\mathbb{R}$, $\lim_{n \to \infty} \frac{x^{2n}}{1 + x^{2n}} \overset{?}{=} ?$

$f_n(-x) = f_n(x)$ sono tutte funzioni PARI

• Se $x = 0$ $\quad \lim_{n \to \infty} \frac{x^{2n}}{1+x^{2n}} = 0 \quad \forall n$

• Se $|x| < 1$
(e $x \neq 0$)

$$\frac{x^{2n}}{1+x^{2n}} = \underbrace{\frac{x^{2n}}{x^{2n}}}_{=1} \cdot \frac{1}{\frac{1}{x^{2n}} + 1} =$$

$$\frac{1}{x^{2n}} \to \begin{cases} 0 & \text{se } |x| > 1 \\ +\infty & \text{se } |x| < 1 \end{cases}$$

se $|x| < 1$

$$\frac{x^{2n}}{1+x^{2n}} = \frac{1}{\underbrace{\left(1 + \frac{1}{x^{2n}}\right)}_{\to +\infty}} \longrightarrow 0$$

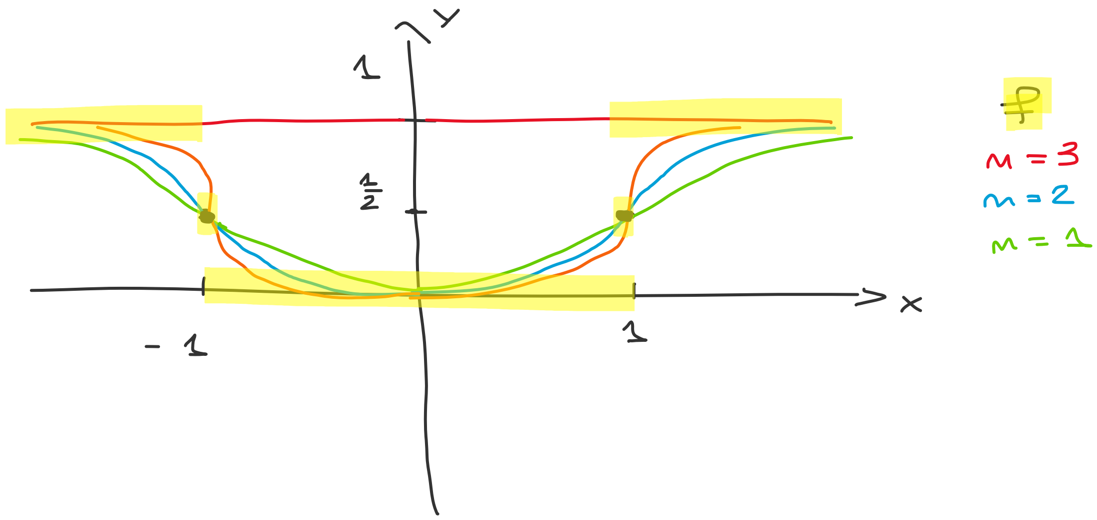
Se $|x| > 1$

$$\frac{x^{2n}}{1+x^{2n}} = \frac{1}{1 + \underbrace{\frac{1}{x^{2n}}}_{\to 0}} \longrightarrow 1$$

Se $|x| = 1$ ovvero se $x = \pm 1$

$$f_n(\pm 1) = \frac{(\pm 1)^{2n}}{1 + (\pm 1)^{2n}} = \frac{1}{2}$$

$$f_n(x) = \frac{x^{2n}}{1+x^{2n}} \xrightarrow{\mathbb{R}} f(x) = \begin{cases} 0 & |x| < 1 \\ \frac{1}{2} & |x| = 1 \\ 1 & |x| > 1 \end{cases}$$



CONV. UNIFORME IN VARI SOTTOINSIEMI DI $\mathbb{R}$

- Studiamo se $f \underset{[-\bar{x},\bar{x}]}{\rightrightarrows} f$ $\quad 0<\bar{x}<x$

Occorre studiare la successione

$$\sup_{[-\bar{x},\bar{x}]} |f_n(x) - f(x)| = L_n \left(= L_n(\bar{x})\right)$$

$$\Downarrow$$

$$\sup_{[-\bar{x},\bar{x}]} f_n(x)$$

$$\frac{d}{dx} f_n(x) = \frac{d}{dx} \frac{x^{2n}}{1+x^{2n}} = \frac{2n\,x^{2n-1}(1+x^{2n}) - x^{2n}\,2n x^{2n-1}}{(1+x^{2n})^2} =$$

$$= \frac{2nx^{2n-1}}{(1+x^{2n})^2} > 0 \iff x > 0$$

$$L_n = \sup_{[-\bar{x},\bar{x}]} f_n(x) = f_n(\bar{x}) \xrightarrow[n\to\infty]{} 0$$

$$\Downarrow$$

$$f_n \rightrightarrows_{[-\bar{x},\bar{x}]} f \qquad 0<\bar{x}<1$$

Ci sarà convergenza uniforme in $[-1,1]$?

$$\sup_{[-1,1]} |f_n(x) - f(x)| = \sup_{]-1,1[} |f_n(x) - f(x)|$$

$$\Downarrow$$

$$f_n(1) = \frac{1}{2} = f(1)$$

$$f_n(-1) = \frac{1}{2} = f(-1)$$

$$= \sup_{]-1,1[} f_n(x) = \sup_{[0,1[} f_n(x) = f_n(1) = \frac{1}{2} \not\to \text{NON converge a ZERO}$$

$\downarrow$ in $[0,1[$ f_n è monotona crescente

NON c'è convergenza uniforme in $[-1,1]$

- Studiamo, ad esempio, la converg. unif. di $(f_n)_n$ in

$(\tilde{x}, +\infty)$, $\tilde{x} > 1$

$$\sup_{(\tilde{x},+\infty)} |f_n(x) - f(x)| = \sup_{(\tilde{x},+\infty)} \left(1 - \frac{x^{2n}}{1+x^{2n}}\right) =$$

$$f(x) = 1$$

$$= \sup_{(\tilde{x},+\infty)} \frac{1}{\underbrace{1+x^{2n}}_{\text{monotona decrescente}}} = \frac{1}{1+(\tilde{x})^{2n}} = 0$$

$$(\tilde{x})^{2n} \to +\infty$$

$$\Downarrow$$

$$f_n \xrightarrow{(\tilde{x},+\infty)} \quad \text{se } \tilde{x} > 1$$

- se $\tilde{x} = 1$. c'é conv unif in $]1, +\infty)$?

$$\sup_{]1,+\infty)} |f_n(x) - f(x)| = \sup_{]1,+\infty)} \frac{1}{\underbrace{1+x^{2n}}_{\text{monotona decrescente}}}$$

$$\Downarrow$$

$$= \frac{1}{2} \not\longrightarrow \text{ NON CONV A ZERO}$$

NON c'é conv unif in $]1, +\infty)$.